Solid abelian groups, II

<u>Plan</u>. 1. Abstract framework + complement

2. Apply it to solidification functor $(-)^{\blacksquare}$.

3. Solid tensor product

4. Proof of the key lemma.

1. <u>Recall</u>. $\mathcal{A}$ ab. cat with colimits

$\mathcal{A}_0 \subset \mathcal{A}$ subcat of some compact proj. objects generating $\mathcal{A}$.

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, bend left, "\subseteq"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{A} \end{tikzcd} \quad \text{such that}$$

$(*)$ $\forall X \in \mathcal{A}_0$, $\forall K = \ker\left(\bigoplus F(\overset{\mathcal{A}_0}{\overset{\cup}{-}}) \to \bigoplus F(\overset{\mathcal{A}_0}{\overset{\cup}{-}})\right)$

then $R\mathrm{Hom}_{\mathcal{A}}(F(X), K) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{A}}(X, K)$

If $(*)$ holds, then

$\mathcal{A}_F \subset \mathcal{A}$ $\qquad\qquad$ $F: \mathcal{A} \longrightarrow \mathcal{A}_F$

$D_F(\mathcal{A}) \subseteq D(\mathcal{A})$ $\qquad\qquad$ $LF: D(\mathcal{A}) \longrightarrow D_F(\mathcal{A})$

$\quad ||$

$D(\mathcal{A}_F)$

$$\mathcal{A}_F = \{ Y \in \mathcal{A} \mid \forall X \in \mathcal{A}_0, \mathrm{Hom}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}(X, Y) \}$$

$$D_F(\mathcal{A}) = \{ C \in D(\mathcal{A}) \mid \forall X \in \mathcal{A}_0, R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C) \}$$

<u>Lemma</u> Consider the condition

$$(*') : \forall X \in A_0,\ \forall C = \quad \cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0$$

where $\quad C_i = \bigoplus F(\underset{\underset{A_0}{\uparrow}}{-}),$

then $\quad RHom(F(X), C) \xrightarrow{\sim} RHom(X, C).$

Then $(*') \Rightarrow (*)$.

<u>Rem.</u> $\quad (*') \xRightarrow{\text{lemma}} (*)$

$(*) \xRightarrow{\cancel{\text{Abstract framework}}} \quad \cancel{A_F \text{ is stable under colimits}}$

$\Rightarrow A_F \supseteq \{ \bigoplus F(\underset{\underset{A_0}{\uparrow}}{-}) \}$

$\Rightarrow$ the above $C \in D(A_F) \Rightarrow (*')$
$\qquad\qquad\qquad\qquad \| \text{ abstract framework}$
$\qquad\qquad\qquad\quad D_F(A)$

$\Rightarrow$ these are equivalent.

2. Our case.

$A = \mathrm{Cond}(Ab)$.

$A_0 = \{ \mathbb{Z}[S] \mid S \text{ extr. disconn.} \}$

$F : A_0 \to A$

$\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{■} \cong \underline{Hom}(\underline{Hom}(\mathbb{Z}[S], \mathbb{Z}), \mathbb{Z})$ functorial

Nat. transf.
$$\begin{tikzcd} \mathbb{Z}[S] \arrow[r, dashed, "\exists !"] & \mathbb{Z}[S]^{■} \\ S \arrow[u] \arrow[ur] & \end{tikzcd}$$

We want to verify

$(*') : \quad \forall S$ extr. disc. $\quad \forall C_\bullet = \cdots \to C_1 \to C_0 \to 0.$

where $C_i = \bigoplus_j \mathbb{Z}[\underset{\text{extr. disc.}}{T_{ij}}]^{\blacksquare}$

Then $R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$
$$\| $$
$$R\Gamma(S, C)$$

Pf. Note that, $C(S, \mathbb{Z})$ = free abelian group for S profinite set.

$\Rightarrow \mathbb{Z}[T_{ij}]^{\blacksquare} = \underline{\mathrm{Hom}}(C(T_{ij}, \mathbb{Z}), \mathbb{Z}) = \prod \mathbb{Z}$.

We may assume $C_\bullet = \cdots \to \bigoplus \prod \mathbb{Z} \to \bigoplus \prod \mathbb{Z} \to 0$.

Notation : $C(S, \mathbb{Z}) = \bigoplus_I \mathbb{Z}$

$$M(S, \mathbb{Z}) = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{Z}) \simeq \prod_I \mathbb{Z}$$
$$M(S, \mathbb{R}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{R}) \simeq \prod_I \mathbb{R}$$
$$M(S, \mathbb{R}/\mathbb{Z}) := \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{R}/\mathbb{Z}) \simeq \prod_I \mathbb{R}/\mathbb{Z}.$$

⇝ exact seq.
$$0 \to M(S, \mathbb{Z}) \to M(S, \mathbb{R}) \to M(S, \mathbb{R}/\mathbb{Z}) \to 0.$$

Key lemma. $\forall S, S'$ profinite, have a nat. iso.

$$\begin{tikzcd}
R\underline{\mathrm{Hom}}(M(S, \mathbb{R}/\mathbb{Z}), C)(S') \arrow[r, phantom, "\simeq"] & R\Gamma(S \times S', C)[-1] \arrow[d, equal] \\
R\underline{\mathrm{Hom}}(\underbrace{M(S, \mathbb{Z})}_{=\mathbb{Z}[S]^{\blacksquare}}, C)(S')[-1] \arrow[u] \arrow[r] & R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)(S')[-1]
\end{tikzcd}$$

Cor. $R\underline{\mathrm{Hom}}(\mathbb{R}, C) = 0$.

pf. $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$

⇝ $R\underline{\mathrm{Hom}}(\mathbb{R}[1], C) \to R\underline{\mathrm{Hom}}(\mathbb{Z}[1], C) \to R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, C)$

Key lemma + $S = \{pt\}$.

S' profinite

$$\begin{tikzcd} \Rightarrow \underline{RHom}(\mathbb{R}/\mathbb{Z}, C)(S') \arrow[r, "\simeq"] & R\Gamma(S', C)[-1] \arrow[d, equal] \\ \underline{RHom}(\mathbb{Z}[1], C)(S') \arrow[u, "2"] \arrow[r, "\sim"] & R\underline{Hom}(\mathbb{Z}[1], C)(S') \end{tikzcd}$$

$\Rightarrow \quad R\underline{Hom}(\mathbb{R}, C) = 0$

Cor. $R\underline{Hom}(\mathbb{Z}[S]^{■}, C) \xrightarrow{\sim} R\underline{Hom}(\mathbb{Z}[S], C)$

Pf.

$$\begin{tikzcd} R\underline{Hom}(M(S,\mathbb{R})[1], C)(S') \arrow[r] \arrow[d, equal] & R\underline{Hom}(\underbrace{M(S,\mathbb{Z})}_{\mathbb{Z}[S]^{■}}[1], C)(S') \arrow[r] & R\underline{Hom}(M(S,\mathbb{R}/\mathbb{Z}), C)(S') \arrow[d, "\wr"] \\ R\underline{Hom}_{\mathbb{R}}(\underbrace{M(S,\mathbb{R})[1]}_{=0}, \underbrace{R\underline{Hom}(\mathbb{R}, C)}_{=0})(S') & & R\underline{Hom}(\mathbb{Z}[S], C)[-1](S') \end{tikzcd}$$

$M(S,\mathbb{R}) \simeq \prod \mathbb{R}$

In particular, $\Rightarrow R\underline{Hom}(\mathbb{Z}[S]^{■}, C) \xrightarrow{\sim} R\underline{Hom}(\mathbb{Z}[S], C)$

this is $(*')$.

Abstract framework applies

$\Rightarrow$. Solid $\subset$ Cond(Ab) abelian subcat stable under lim, colim, ext.

has compact proj. generators $\mathbb{Z}[S]^{■}$, S extr. disconn.

has a left adjoint $(-)^{■}$.

$$\begin{tikzcd} D(\mathrm{Solid}) \arrow[r, phantom, "\subset"] & D(\mathrm{Cond}(\mathrm{Ab})) \arrow[l, bend left, "(-)^{L■}"] \end{tikzcd}$$

$D(\mathrm{Solid}) = \{ C \in D(\mathrm{Cond\,Ab}) \mid H^i(C) \in \mathrm{Solid} \}$.

Cor (i) $\mathbb{R}^{L\blacksquare} = 0$.

(ii). $\forall C \in D(\mathrm{Solid})$. $\forall S$ profinite.

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C).$$

(iii). $\{$ compact proj. objects of Solid $\}$

$$\|$$

$\{ \prod_I \mathbb{Z}$ for varying set $I \}$

pf. (i). By definition,

$$\forall C \in D(\mathrm{Solid}). \quad \underset{D(\mathrm{Solid})}{R\underline{\mathrm{Hom}}}(\mathbb{R}^{L\blacksquare}, C) \overset{\mathrm{def}}{=} \underset{D(\mathrm{CondAb})}{R\underline{\mathrm{Hom}}}(\mathbb{R}, C).$$

It is enough to show that $R\underline{\mathrm{Hom}}(\mathbb{R}, C) = 0$.

- if $C = \cdots \to \bigoplus_j \mathbb{Z}[T_j]^{\blacksquare} \to \cdots \to 0$ (seen)

  $\Rightarrow$ if C is bounded from above ✓.

- By naive truncation on the right $\Rightarrow$ C unbounded.
  ok for.

(ii). similar.

(iii). $\mathbb{Z}[S]^{\blacksquare}$, S extr. disconn. are compact proj in Solid.

$\|$ Specker

$\prod_I \mathbb{Z}$

- For any set J. $\prod_J \mathbb{Z}$ is compact proj.

  for sufficiently "big" S. extr. disconn.

$$\underset{\text{direct summand}}{\bigoplus_J \mathbb{Z} \hookrightarrow C(S, \mathbb{Z}) \simeq \bigoplus_{I \text{ big}} \mathbb{Z}}$$

$\Rightarrow \prod_J \mathbb{Z}$ is a direct summand of $\mathbb{Z}[S]^\blacksquare$ compact proj.

$\Rightarrow \prod_J \mathbb{Z}$ is compact proj.

- Let $P \in$ Solid be compact projective

$$\begin{tikzcd}
\bigoplus \prod \mathbb{Z} \arrow[r, two heads] & P \\
 & P \arrow[u, "\mathrm{id}"'] \arrow[ul, "\exists"]
\end{tikzcd}$$

proj.

$\xrightarrow{\text{compact}}$ this factors through $\bigoplus_{\text{finite}} \prod \mathbb{Z} \cong \prod \mathbb{Z}$

$\Rightarrow$ P is a direct summand of $\prod \mathbb{Z}$

$\Rightarrow P \cong \prod \mathbb{Z}$.

$$\left(\begin{array}{l} \underline{\mathrm{Hom}}(\prod_I \mathbb{Z}, \mathbb{Z}) = \bigoplus_I \mathbb{Z} \\ \underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z}, \mathbb{Z}) = \prod_I \mathbb{Z} \\ \text{a direct summand of } \bigoplus \mathbb{Z} \text{ is } \bigoplus \mathbb{Z} \end{array} \right)$$

3. Solid tensor product.

<u>Thm/Def</u>. (i). $\exists!\ -\otimes^\blacksquare-$ on Solid st. $(-)^\blacksquare : \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Solid}$ is symmetric monoidal.

(ii). $\exists!\ -\otimes^{L\blacksquare}-$ on $D(\mathrm{Solid})$ st. $(-)^{L\blacksquare}$ is sym. monoidal.

(iii). $-\otimes^{L\blacksquare}- = L(-\otimes^\blacksquare-)$

Pf. Proof of (ii). is similar to (i).

(i) The sym. monoidal $\Rightarrow$ $M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}$, $M, N \in$ Solid.

Have to verify: $(M \otimes N)^{\blacksquare} \xrightarrow{\sim} M^{\blacksquare} \otimes^{\blacksquare} N^{\blacksquare} = (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}$
for $M, N \in \mathrm{Cond}(\mathrm{Ab})$

$$\begin{tikzcd}
(M \otimes N)^{\blacksquare} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare} \\
& (M^{\blacksquare} \otimes N)^{\blacksquare} \arrow[ur] &
\end{tikzcd}$$

Since $- \otimes -$, $(-)^{\blacksquare}$ are left adj. functors,

$\Rightarrow$ comm. with colim

$\Rightarrow$ may assume

$$M = \mathbb{Z}[S], \quad N = \mathbb{Z}[T]. \qquad S, T \text{ extr. disconn.}$$

$\forall X \in$ Solid,

$$\begin{array}{ccc}
\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T], X) & \xrightarrow{\sim} & \mathrm{Hom}(\mathbb{Z}[S] \otimes \mathbb{Z}[T], X) \\
\| & & \| \\
\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, X)(T) & \xrightarrow{\sim} & \mathrm{Hom}(\mathbb{Z}[S], X)(T)
\end{array}$$

(iii). May assume $M = \mathbb{Z}[S]^{\blacksquare}$, $N = \mathbb{Z}[T]^{\blacksquare}$ $\in$ Solid

$$\begin{array}{ccc}
M \otimes^{L\blacksquare} N & = & (M \otimes^{\blacksquare} N)[0]. \\
 & & \| \\
\| & & (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T]^{\blacksquare})^{\blacksquare} \\
 & & \| \\
(\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{L\blacksquare} & \xrightarrow[\text{last time}]{\sim} & (\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{\blacksquare}
\end{array}$$

$\underline{\mathbb{Z}[S \times T]}$

Prop. / Ex $\displaystyle \prod_I \mathbb{Z} \otimes^{L\blacksquare} \prod_J \mathbb{Z} = \Big(\prod_{I \times J} \mathbb{Z} \Big)[0]$

pf. $\underline{\mathrm{Hom}}(\bigoplus_I \mathbb{Z}, \mathbb{Z}) \otimes^{L\blacksquare} \underline{\mathrm{Hom}}(\bigoplus_J \mathbb{Z}, \mathbb{Z}) \xrightarrow{\sim} \underline{\mathrm{Hom}}((\bigoplus_I \mathbb{Z}) \otimes (\bigoplus_J \mathbb{Z}), \mathbb{Z})$.

May embed : $\bigoplus_I \mathbb{Z} \hookrightarrow C(S, \mathbb{Z})$

S, T extr. disconn.

$\bigoplus_J \mathbb{Z} \hookrightarrow C(T, \mathbb{Z})$

as direct summands.

Reduce to the case of

$C(S, \mathbb{Z}) \otimes C(T, \mathbb{Z})$
$\shortparallel$
$C(S \times T, \mathbb{Z})$

$$\mathbb{Z}[S]^{\blacksquare} \otimes^{L\blacksquare} \mathbb{Z}[T]^{\blacksquare} \xrightarrow{\sim} \mathbb{Z}[S \times T]^{\blacksquare}$$

$\square$.

Ex. $\mathbb{Z}[[u]] \otimes^{L\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}[[u, T]]$

p prime, $\mathbb{Z}_p = [\, 0 \to \underline{\mathbb{Z}[[u]]} \xrightarrow{\times(u-p)} \underline{\mathbb{Z}[[u]]} \to 0 \,]$

$\mathbb{Z}[[u]] \downarrow \underline{\mathbb{Z}_p}$

$\underline{\mathbb{Z}/p^n}$

$\Rightarrow \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}_p[[T]]$

$$\Rightarrow \mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_\ell = \begin{cases} 0, & p \neq \ell. \\ \mathbb{Z}_p, & \ell = p. \end{cases}$$

4. Proof of the key lemma.

$C_\bullet = \quad \cdots \to C_1 \to C_0 \to 0 \quad , \quad C_i = \bigoplus \prod \mathbb{Z}.$

Then $\forall S, S'$ profinite,

$$\underline{R\mathrm{Hom}}\big(\underset{\substack{\wr \\ \prod \mathbb{R}/\mathbb{Z}}}{M(S, \mathbb{R}/\mathbb{Z})}, C\big)(S') \simeq R\Gamma(S \times S', C)[-1]$$

pf. Case 1 C bounded $\overset{\text{reduce}}{\rightsquigarrow}$ $C = \big(\bigoplus_{i \in I} \prod_{j \in I_i} \mathbb{Z}\big)[0]$

Recall $M \in \mathrm{Cond}(\mathrm{Ab})$ is called pseudocoherent.

if $\mathrm{Ext}^i(M,-)$ commutes with all filtered colimits.

equiv : $\exists$ resolution : $\to \underset{\text{finite}}{\bigoplus} \mathbb{Z}[\underset{\text{extr. discon.}}{S}] \to \cdots \to \underset{\text{finite}}{\bigoplus} \mathbb{Z}[S] \to M \to 0$

- X compact Hausdorff space $\Rightarrow \mathbb{Z}[X]$ pseudocoherent

  ($\underset{\text{extr. discon}}{S_\bullet} \to X$ hypercover)

- $\forall X \in \mathrm{Cond}(\mathrm{Ab})$, Breen – Deligne

  $\Rightarrow \cdots \underset{\text{finite}}{\bigoplus} \mathbb{Z}[X^{r_i}] \to \cdots \to \mathbb{Z}[X] \to X \to 0$

  $\Rightarrow$ Any compact Hausdorff abelian group is pseudo coherent. (in Cond Ab)

<u>Ex</u>. $\left.\begin{array}{l} M(S, \mathbb{R}/\mathbb{Z}) \simeq \prod_I \mathbb{R}/\mathbb{Z}. \quad S \text{ profinite} \\ \mathbb{Z}[S]. \\ M(S, \mathbb{R}/\mathbb{Z}) \otimes \mathbb{Z}[S']. \quad S' \text{ profinite} \end{array}\right\}$ pseudocoherent

$$\boxed{\underline{R\mathrm{Hom}}(\underset{\prod \mathbb{R}/\mathbb{Z}}{\underset{\wr}{M(S, \mathbb{R}/\mathbb{Z})}}, C)(S') \simeq R\Gamma(S \times S', C)[-1]}$$

Back to the proof. S, S' profinite.

$\underline{R\mathrm{Hom}}(M(S, \mathbb{R}/\mathbb{Z}), \bigoplus \prod \mathbb{Z})(S')$

$= R\mathrm{Hom}(\underline{M(S, \mathbb{R}/\mathbb{Z}) \otimes \mathbb{Z}[S']}, \bigoplus \prod \mathbb{Z})$

$$\overset{\text{pseudo-coherent}}{=} \bigoplus \prod \mathrm{RHom}(M(S, \mathbb{R}/\mathbb{Z}) \otimes \mathbb{Z}[S'], \mathbb{Z})$$

$$= \bigoplus \prod \underline{\mathrm{RHom}}(\underbrace{\prod_I \mathbb{R}/\mathbb{Z}, \mathbb{Z}}_{\substack{\| \\ (\bigoplus_I \mathbb{Z})[-1] \\ \| \\ C(S,\mathbb{Z})[-1]}})(S')$$

$$\Rightarrow \mathrm{LHS} = \bigoplus \prod C(S \times S', \mathbb{Z})[-1]$$

- $$\mathrm{RHS} = R\Gamma(S \times S', \bigoplus \prod \mathbb{Z}) \overset{\text{pseudocoh}}{=} \bigoplus \prod R\Gamma(S \times S', \mathbb{Z})$$

$$R\Gamma(S \times S', \bigoplus\prod\mathbb{Z}) = \mathrm{RHom}(\mathbb{Z}[S \times S'], \bigoplus \prod \mathbb{Z})$$

$$\bigoplus \prod R\Gamma(S \times S', \mathbb{Z}) = \bigoplus \prod C(S \times S', \mathbb{Z})[0]$$

Case 2. C unbounded. (on the left)

$$\boxed{\underline{\mathrm{RHom}}(\underset{\substack{\| \\ \prod \mathbb{R}/\mathbb{Z}}}{M(S, \mathbb{R}/\mathbb{Z})}, C)(S') \simeq R\Gamma(S \times S', C)[-1]}$$

- Using naive truncation: $C_{\leq i} = \to 0 \to C_i \to C_{i-1} \to \cdots$

$$\begin{array}{c} C_{\leq i} \\ \downarrow \\ C \\ \downarrow \\ C_{\geq i+1} \end{array}$$

$\rightsquigarrow$ enough to show that if C is replaced by $C_{\geq i+1}$,

then the two sides are concentrated in

homological deg. $\geq i, i-1, i-2$.

$\rightsquigarrow$ enough to show that

$R\underline{\mathrm{Hom}}(M(S, \mathbb{R}/\mathbb{Z}), C)$, $R\Gamma(S, C)$ are concentrated

in coh. deg ≤ 1.

- Need to show ________ for $\tau_{\leq i} C$.

$$\begin{tikzcd}
C \arrow[d] = & \to C_{i+1} \arrow[r, "d_i"] & C_i \arrow[r] & C_{i-1} \arrow[r] & \cdots & \\
\tau_{\leq i} C = & \to 0 \arrow[r] & \operatorname{coker} d_i \arrow[r] & C_{i-1} \arrow[r] & \cdots & \boxed{\text{bounded}} \\
0 \to \underline{\ker d_i} \arrow[r] & C_{i+1} \arrow[r, "d_i"] & C_i \arrow[u, "\simeq \text{ q.iso}"] \arrow[r] & \underline{C_{i-1}} \arrow[r] & \cdots & \boxed{\text{bounded}}
\end{tikzcd}$$

Have to see: $R\underline{\mathrm{Hom}}(M(S, \mathbb{R}/\mathbb{Z}), \ker d_i)$, $R\Gamma(S, \underline{\ker d_i})$

concentrated in deg ≤ 1.

$$\begin{tikzcd}
\oplus \prod \mathbb{Z} \arrow[r, "d_i"] \arrow[d, bend right] & \oplus \prod \mathbb{Z} \\
\prod \mathbb{Z} \arrow[r, "d_i"] & \prod \mathbb{Z}
\end{tikzcd}$$

$\ker d_i$ is not easy to describe

$$\oplus \mathbb{Z} \to \oplus \mathbb{Z} \to \underline{?} \to 0$$

We will use $\mathbb{R}$. $\mathbb{R}/\mathbb{Z}$.

<u>Observation</u> if.

$$\begin{tikzcd}
\prod \mathbb{Z} \arrow[r, "d_i"] \arrow[d, phantom, "\cap"] & \prod \mathbb{Z} \arrow[d] \\
\prod \mathbb{R} \arrow[r, "\exists ! d_{\mathbb{R}, i}"] \arrow[d] & \prod \mathbb{R} \arrow[d] \\
\prod \mathbb{R}/\mathbb{Z} \arrow[r, "d_{\mathbb{R}/\mathbb{Z}, i}"] & \prod \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

$\ker d_{\mathbb{R}/\mathbb{Z}, i}$ = compact ab. group.

$\ker d_{\mathbb{R},i}$?

$$\begin{array}{ccccccc}
\prod \mathbb{Z} & \xrightarrow{d_i} & \prod \mathbb{Z} & \xleftrightarrow{\underline{\mathrm{Hom}}(-,\mathbb{Z})} & \bigoplus \mathbb{Z} & \xrightarrow{\delta^i} & \bigoplus \mathbb{Z} \\
\cap & & \cap & & \cap & & \cap \\
\prod \mathbb{R} & \xrightarrow{d_{\mathbb{R},i}} & \prod \mathbb{R} & \xleftarrow{\underline{\mathrm{Hom}}_{\mathbb{R}}(-,\mathbb{R})} & \bigoplus \mathbb{R} & \xrightarrow{\delta^i_{\mathbb{R}}} & \bigoplus \mathbb{R}
\end{array}$$

map of $\mathbb{R}$-vector spaces

$\Rightarrow \operatorname{coker} \delta^i_{\mathbb{R}} = \bigoplus \mathbb{R}$

$\Rightarrow \ker d_{\mathbb{R},i} = \prod \mathbb{R}$.

Summerize the proof.

- $\underline{\text{extend } C \text{ to } C_{\mathbb{R}}}$

$$\begin{array}{ccc}
C & & C_{\mathbb{R}} \\
\| & & \| \\
\to \bigoplus \prod \mathbb{Z} \to \cdots & & \to \underline{\bigoplus} \prod \mathbb{R} \to \cdots
\end{array}$$

$\rightsquigarrow \quad 0 \to C \to C_{\mathbb{R}} \to C_{\mathbb{R}/\mathbb{Z}} \to 0.$

$\rightsquigarrow$ enough to prove

$\underline{R\mathrm{Hom}}\left(M(S,\mathbb{R})/\mathbb{Z}, \begin{array}{c}\tau_{\leq i} C_{\mathbb{R}} \\ \tau_{\leq i} C_{\mathbb{R}/\mathbb{Z}}\end{array}\right)$ concentrated in

$R\Gamma\left(S, \begin{array}{c}\tau_{\leq i} C_{\mathbb{R}} \\ \tau_{\leq i} C_{\mathbb{R}/\mathbb{Z}}\end{array}\right)$ deg ≤ 1.

$\rightsquigarrow$ breaks into $\ker d_{\mathbb{R},i}$ $\leftarrow \underline{\dim} \prod \mathbb{R}$

$\ker d_{\mathbb{R}/\mathbb{Z},i}$

$\bigoplus \prod \mathbb{R}$, $\bigoplus \prod \mathbb{R}/\mathbb{Z}$

using pseudocoherence of $\mathbb{Z}[S]$, $M(S,\mathbb{R})$

$\rightsquigarrow$ $\operatorname{colim} \underline{R\mathrm{Hom}}(M(S,\mathbb{R}), \prod \mathbb{R}/\mathbb{Z})$ compact abelian group

$\prod\mathbb{T}$

$\operatorname{colim} R\Gamma(S, \prod\mathbb{R}$ compact ab. group$)$

concent. in deg ≤ 1

Talk 4:

Talk 3

CQFD

It remains to see $\bigoplus \prod \mathbb{Z} \xrightarrow{d_i} \bigoplus \prod \mathbb{Z}$

$$\begin{array}{ccc} \bigoplus\prod\mathbb{Z} & \xrightarrow{d_i} & \bigoplus\prod\mathbb{Z} \\ \cap & & \cap \\ \bigoplus\prod\mathbb{R} & \overset{\exists!}{\dashrightarrow} & \bigoplus\prod\mathbb{R} \end{array}$$

enough to show that

$$\mathrm{Hom}(\prod\mathbb{Z}, \bigoplus\prod\mathbb{R}) \xleftarrow{\sim} \mathrm{Hom}(\prod\mathbb{R}, \bigoplus\prod\mathbb{R})$$

enough to show that

$$R\mathrm{Hom}(\prod \mathbb{R}/\mathbb{Z}, \bigoplus\prod\mathbb{R}) = 0$$

But: $\prod \mathbb{R}/\mathbb{Z}$ compact ab. group $\Rightarrow$ pseudocoherent

$$\Rightarrow \text{LHS} = \bigoplus \prod \underline{R\mathrm{Hom}(\prod \mathbb{R}/\mathbb{Z}, \mathbb{R})} = 0$$

Ex. X CW complex. $\mathbb{Z}[X]^{L\blacksquare} \cong H_\bullet(X)$

Rem. ((Clausen, mathoverflow)).

M pseudocoherent, then $M^{L\blacksquare} = R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(M,\mathbb{Z}),\mathbb{Z})$
$\in \mathrm{Cond}(\mathrm{Ab})$ "reduce to $M = \mathbb{Z}[S]$"

Morally. $M \simeq [\cdots \to \bigoplus_{\text{finite}} \mathbb{Z}[S] \to \bigoplus_{\text{finite}} \mathbb{Z}[S] \to 0]$

$\Big\}$ double dual against $\mathbb{Z}$

$\underline{\mathrm{Hom}}(\underbrace{\mathbb{Z}[S]^{\blacksquare}}_{\substack{\shortparallel \\ \prod \mathbb{Z}}}, \mathbb{Z})$

$M^{L\blacksquare} = [\ \to \bigoplus_{\text{finite}} \mathbb{Z}[S]^{\blacksquare} \to \bigoplus_{\text{finite}} \mathbb{Z}[S]^{\blacksquare} \to 0\ ]$

$\simeq \bigoplus \mathbb{Z}$

X finite CW-complex. $\Rightarrow$ compact Hausdorff

$$R\mathrm{Hom}(\mathbb{Z}[X], \mathbb{Z}) = R\Gamma_{\mathrm{cond}}(X, \mathbb{Z}) \overset{\downarrow}{=} R\Gamma(X, \mathbb{Z})$$

$$= R\mathrm{Hom}(\underset{\sim}{H_\bullet(X)}, \mathbb{Z})$$

finite cplx.
finite free ab. groups

$$\Rightarrow \mathbb{Z}[X]^{L\blacksquare} \simeq H_\bullet(X).$$

$$R\underline{\mathrm{Hom}}(\ 0 \to \bigoplus \mathbb{Z} \to \bigoplus \mathbb{Z} \to \cdots\ ,\ \mathbb{Z})$$

$\wr ?$

$$\cdots \to \prod \mathbb{Z} \to \prod \mathbb{Z} \to 0$$